$$(1+x)^N = \sum_{k=0}^{N} \binom{N}{k} x^k$$ ← Binomial Theorem

take derivative

$$\boxed{N(1+x)^{N-1} = \sum_{k=0}^{N} k \binom{N}{k} x^{k-1}}$$

↑ can plug in $x=1$, $x=-1$, $x=2$

⋮

and more fun facts.

A prime number is an integer >1 such that its only divisors are 1 and itself. Give the first 20 primes.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29,
31 ...

algorithm: Sieve of ______?

2, 3, ~~4~~, 5, ~~6~~, 7, ~~8~~, ~~9~~, ~~10~~, 11, ~~12~~, 13, ~~14~~, ~~15~~
~~16~~, 17, ~~18~~, 19, ~~20~~, ~~21~~, ~~22~~, 23, ~~24~~, ~~25~~, ~~26~~, ~~27~~, ~~28~~
29, ~~30~~, 31, ~~32~~, ~~33~~, ~~34~~, ~~35~~, ~~36~~, 37, ~~38~~, ~~39~~, ~~40~~
41, ~~42~~, 43, ~~44~~, ~~45~~, ~~46~~, 47, ~~48~~, ~~49~~, ~~50~~, ~~51~~, ~~52~~,
53, ~~54~~, ~~55~~, ~~56~~, ~~57~~, ~~58~~, 59, ~~60~~, 61, ~~62~~, ~~63~~, ~~64~~,
~~65~~, 67, ~~69~~, 71, 73, ~~75~~, ~~77~~, 79, ~~81~~, 83, ~~85~~,
~~87~~, 89, ~~91~~, ~~93~~, ~~95~~, 97, ~~99~~

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41,
43, 47, 53, 59, 61, 67, 71, 73, 79, 83,

89, 97

More examples of recursive sequences.

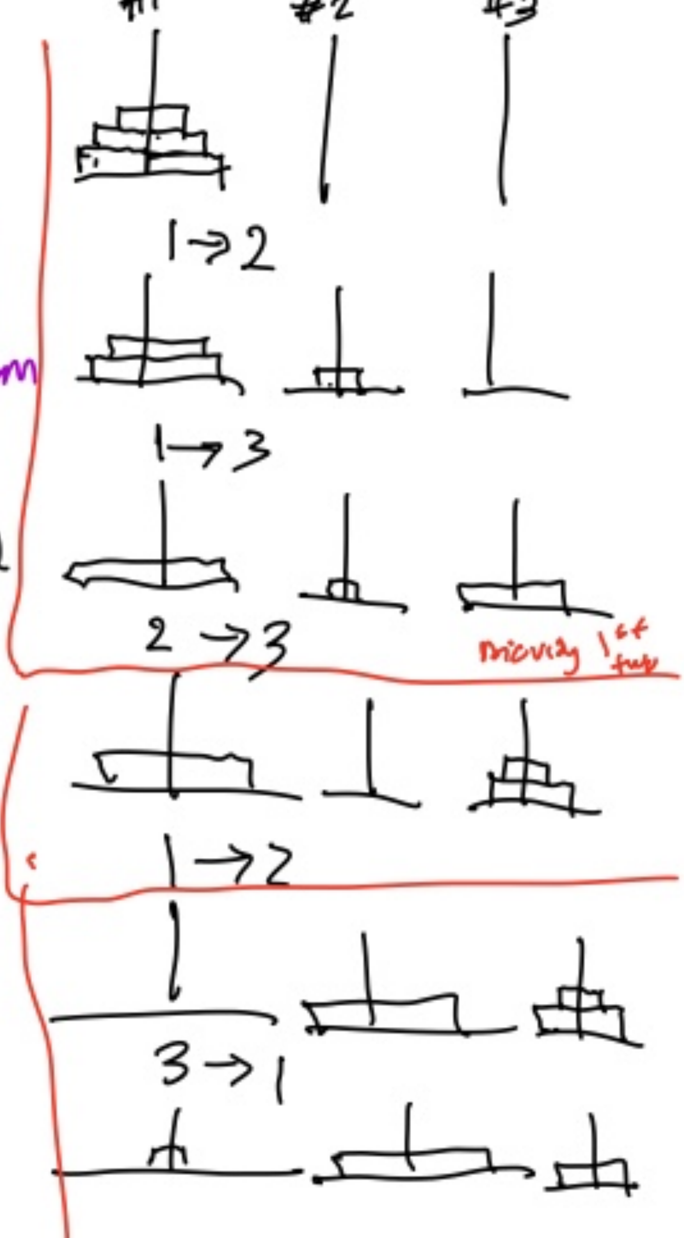
① Game – Tower of Hanoi

3 pegs

Moves: Take a disk, place it on a different peg – But you are only allowed to do it if there are no smaller disks on that peg.

Goal: After a sequence of moves, transfer all of the disks from #1 peg to #2 peg

Example (N = 3)



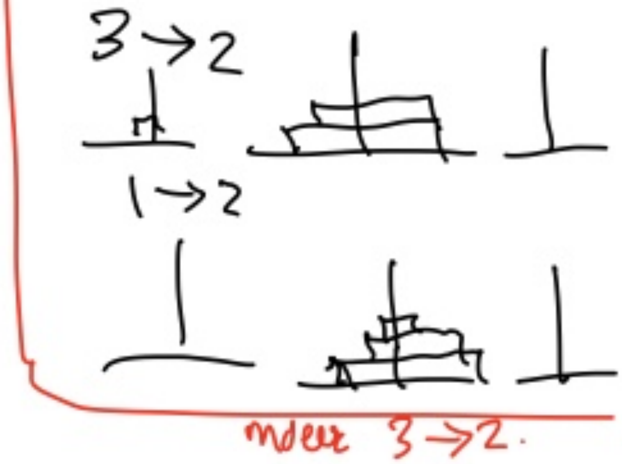
H_N = # of moves required to finish the game (minimally)

$H_1 = 1$

$H_2 = 3$

$H_3 = 7$

Is there a recursive formula for H_N in terms of previous H_{N-1}, H_{N-2}.



Our answer

$$H_N = \underbrace{H_{N-1}}_{\text{move all but bottom to 3rd peg.}} + \underbrace{1}_{\text{bottom to 2nd peg}} + \underbrace{H_{N-1}}_{\text{move all on 3rd peg to 2nd peg}}$$

$$\boxed{\begin{aligned} H_N &= 2H_{N-1}+1 \\ H_1 &= 1 \end{aligned}}$$

Recursive formula.

$H_1 = 1$

$H_2 = 2\cdot 1+1 = 3$

$H_3 = 2\cdot 3+1 = 7$

$H_4 = 2\cdot 7+1 = 15$

$H_5 = 2\cdot 15+1 = 31$

$H_6 = 2\cdot 31+1 = 63$

$2\cdot 2^{N-1}-1$

closed form formula.

$$\boxed{H_N = 2^N-1}$$

(We could also prove this formula by induction.)

Proof: Base case $N=1$: $H_1 = 1$ given

formula $2^1 - 1 = 1$. ✓

$\therefore$ $N=1$ case is true.

Now, assume that $H_k = 2^k - 1$ for some $k \in \mathbb{N}$.

Then $H_{k+1} = 2 \cdot H_k + 1$

using recursion formula

$= 2 \cdot (2^k - 1) + 1$

$= 2^{k+1} - 2 + 1$

$= 2^{k+1} - 1$.

$\therefore$ The formula is true for $N = k+1$.

By induction, $H_N = 2^N - 1$ for all $N \in \mathbb{N}$.

BTW: If we play the same game with 4 pegs, noone knows how to come up with a formula for H_n.

$\boxed{\text{Example}}$ $n! = n(n-1)(n-2)\cdots 1 = \text{Fact}(n)$.

Recursive formula for $n!$

$$\begin{cases} \text{Fact}(1) = 1 \\ \text{Fact}(n) = n\,\text{Fact}(n-1) \end{cases}$$

Is there a closed form formula?